

CURVE CLASSES ON RATIONALLY CONNECTED VARIETIES

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ABSTRACT. This note proves every curve on a rationally connected variety is algebraically equivalent to a \mathbb{Z} -linear combination of rational curves.

1. INTRODUCTION

In [7] and [10], the following question is asked by Professor János Kollár and Professor Claire Voisin:

1.1. Question. *For a smooth projective rationally connected variety over \mathbb{C} with dimension n , is every integral Hodge $(n-1, n-1)$ -class a \mathbb{Z} -linear combination of cohomology classes of rational curves?*

This question can be separated into two questions, as in [10]:

- For a smooth projective rationally connected variety over \mathbb{C} , is every integral hodge $(n-1, n-1)$ -class a \mathbb{Z} -linear combination of cohomology classes of curves?
- For a smooth projective rationally connected variety over \mathbb{C} , is every curve class a \mathbb{Z} -linear combination of cohomology classes of rational curves?

While generally unknown, the dimension 3 case of the first question is implied by the following result of Professor Claire Voisin:

1.2. Theorem. [9] *For a smooth projective 3-fold which is uniruled or Calabi-Yau, every integral Hodge $(2, 2)$ -class is a \mathbb{Z} -linear combination of cohomology classes of curves.*

We will solve the second question in this note, resulting in the following main theorem:

1.3. Theorem. *Let X be a smooth projective rationally connected variety over \mathbb{C} then every curve on X is algebraically equivalent to a \mathbb{Z} -linear combination of rational curves.*

The idea of proof is to first lift any irreducible curve C in X to $X \times \mathbb{P}^1$, by result of [5], there are very free rational curves which are horizontal with respect to the projection to \mathbb{P}^1 , just adding enough of these curves to form a comb– which can be smoothed to a new horizontal curve \hat{C} with $H^1(\hat{C}, N_{\hat{C}}) = 0$ and which is "flexible" in the sense of [5], namely, the map

$$\bar{M}_{g,0}(X \times \mathbb{P}^1, \beta) \rightarrow \bar{M}_{g,0}(\mathbb{P}^1, d)$$

as defined in [2] will be proper and surjective at the point represented by the curve \hat{C} , where $g(\text{resp } \beta, d.)$ is the genus (resp cohomology class, degree over \mathbb{P}^1 .) of \hat{C} — this map will contract non-stable components which emerges after projection to \mathbb{P}^1 , all candidancies of which are rational curves with at most 2 marked points or elliptic curves with no marked points, so we can exclude the case of elliptic curves by recalling that deformation of \hat{C} will again be connected; and the hurwitz scheme $\bar{M}_{g,0}(\mathbb{P}^1, d)$ is irreducible, so just degenerating the image of \hat{C} to a sum of rational curves, by surjectivity and what we marked before about the non-stable components, we will get a sum of rational curves in $\bar{M}_{g,0}(X \times \mathbb{P}^1, \beta)$ which is algebraically equivalent to \hat{C} , simply fushing forward this relation back to X , we will get the result.

Recalling that algebraic equivalence implies cohomological equivalence and combining theorem 1.2, we have:

1.4. Corollary. *For a smooth projective rationally connected 3-fold, every integral Hodge $(2, 2)$ -class is a \mathbb{Z} -linear combination of cohomology classes of rational curves.*

In an upcoming paper [8] of the author with Zhiyu Tian, we will try to explore further application of the "trivial product" trick in the proof.

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2. PRELIMINARIES

2.5. Definition. *Let X be smooth projective variety over \mathbb{C} . It is rationally connected if there is a rational curve passing through 2 general points of X . By a free (resp.very free) curve in X we mean a rational curve $C \subset X$ with $T_X|_C$ non-negative (resp.ample). It is well-known that X to be rationally connected is equivalent to the existence of very free curves on X .*

2.6. Definition. *Let C be a connected nodal curve, X any variety, we call a map $f : C \rightarrow X$ a stable map if every component of C which is mapped to constant are either:*

- *A curve with arithmetic genus > 1*
- *A curve with arithmetic genus 1 with at least 1 nodal point.*
- *A curve with arithmetic genus 0 with at least 3 nodal points.*

It is well-known that we have good compactified moduli stack of all stable maps $f : C \rightarrow X$, $\bar{M}_{g,0}(X, \beta)$ and β is the cohomology class of C in X , for reference see [4].

2.7. Definition. Let k be an arbitrary field. A comb with n teeth over k is a projective curve with $n + 1$ irreducible components C_0, C_1, \dots, C_n over \bar{k} satisfying the following conditions:

- 1.) The curve C_0 is defined over k .
- 2.) The union $C_1 \cup \dots \cup C_n$ is defined over k . (Each individual curve may not be defined over k .)
- 3.) The curves C_1, \dots, C_n are smooth rational curves disjoint from each other, and each of them meets C_0 transversely in a single smooth point of C_0 (which may not be defined over k).

The curve C_0 is called the handle of the comb, and C_1, \dots, C_n are called the teeth. A rational comb is a comb whose handle is a smooth rational curve.

3. PROOF OF THE MAIN THEOREM

Let Y be a smooth projective variety with a morphism $Y \rightarrow \mathbb{P}^1$ whose general fibers are rationally connected.

For a class $\beta \in H_2(Y, \mathbb{Z})$ having intersection number d with a fiber of the map π . We have then a natural morphism as in [2]:

$$\varphi : \bar{M}_{g,0}(Y, \beta) \rightarrow \bar{M}_{g,0}(\mathbb{P}^1, d)$$

defined by composing a map $f : C \rightarrow Y$ with π and collapsing components of C as necessary to make the composition πf stable.

3.8. Lemma. For a stable map $f : C \rightarrow Y$ which is non-constant, the components that are contracted under

$$\varphi : \bar{M}_{g,0}(Y, \beta) \rightarrow \bar{M}_{g,0}(\mathbb{P}^1, d)$$

are all rational.

Proof. By Definition 2.6, the only possible non-stable components are:

- Smooth rational curve with at most 2 intersection points with other components of C
- Nodal rational curve with at most 1 intersection point with other components of C
- Smooth elliptic curve which is a connected component of the curve contracted at some step.

The third case can be excluded since C will always be connected after contraction. \square

3.9. Definition. Let $f : C \rightarrow Y$ be a stable map from a nodal curve C of genus g to Y with class $f_*[C] = \beta$. We say that f is flexible relative to π if the map $\varphi : \bar{M}_{g,0}(Y, \beta) \rightarrow \bar{M}_{g,0}(\mathbb{P}^1, d)$ is dominant at the point $[f] \in \bar{M}_{g,0}(Y, \beta)$ and $\pi : C \rightarrow \mathbb{P}^1$ is flat.

3.10. Proposition. A flexible curve $f : C \rightarrow Y$ can be degenerated to an effective sum of rational curves in Y .

Proof. It is a classical fact that the variety $\bar{M}_{g,0}(\mathbb{P}^1, d)$ has a unique irreducible component whose general member corresponds to a flat map $f : C \rightarrow \mathbb{P}^1$, see [3]. Since the map $\varphi : \bar{M}_{g,0}(Y, \beta) \rightarrow \bar{M}_{g,0}(\mathbb{P}^1, d)$ is proper, and $\pi : C \rightarrow Y$ is flexible then φ will be surjective on the component of $\pi : C \rightarrow Y$. By Lemma 3.8 it is enough to find a degeneration of $C \rightarrow \mathbb{P}^1$ in $\bar{M}_{g,0}(\mathbb{P}^1, d)$ as a sum of rational curves, which is elementary. □

3.11. Lemma. [1] *Let X be a smooth projective variety of dimension at least 3 over an algebraically closed field. Let $D \subset X$ be a smooth irreducible curve and M a line bundle on D . Let $C \subset X$ be a very free rational curve intersecting D and let \hat{C} be a family of rational curves on X parametrized by a neighborhood of $[C]$ in $\text{Hom}(\mathbb{P}^1, X)$.*

Then there are curves $C_1, \dots, C_p \in \hat{C}$ such that $D^ = D \cup C_1 \cup \dots \cup C_p$ is a comb and satisfies the following conditions:*

- 1.) *The sheaf N_{D^*} is generated by global sections.*
- 2.) *$H^1(D^*, N_{D^*} \otimes M^*) = 0$, where M^* is the unique line bundle on D^* that extends M and has degree 0 on the C_i .*

Which leads to the following:

3.12. Lemma. *For any curve C in a rationally connected variety X , and any $m \gg 0$ free curves C_1, \dots, C_m , such that $C \cup C_1 \cup C_2 \dots \cup C_m$ is a comb as in Definition 2.7, then: There is a sub-comb $C \cup C_{i_1} \cup C_{i_2} \dots \cup C_{i_k}$, $k \leq m$ which can be deformed to an irreducible curve C' with $H^1(C', N_{C'}) = 0$, where $N_{C'}$ is the normal bundle of C' .*

3.13. Remark. We note that C can be highly singular in X , but let C' be the normalization of C , embed it as $C' \rightarrow \mathbb{P}^3$ and then project a small deformation of the diagonal map $C' \rightarrow X \times \mathbb{P}^3$ to X —we can get a deformation of C as a smooth sub-curve $C' \subset X$, then we can apply Lemma 3.11 to get Lemma 3.12.

3.14. Theorem. (Main Construction of [5]) *For any multisection $B \rightarrow \mathbb{P}^1$, there are rational curves C_i 's such that $B \cup C_1 \cup \dots \cup C_m$ can be deformed to a flexible curve of $Y \rightarrow \mathbb{P}^1$.*

Now we can prove our main theorem.

Proof. Take $Y = X \times \mathbb{P}^1$, for any irreducible curve $C \subset X$, lift it to a curve C' in $X \times 0 \subset Y$. Since Y is rationally connected, we can add enough free curves of Y which are horizontal with respect to the projection $Y \rightarrow \mathbb{P}^1$, such that the comb can be deformed by Lemma 3.12 to a multisection M of the fibration $Y \rightarrow \mathbb{P}^1$. Then by Theorem 3.14, we can add some other rational curves to M to be deformed to a flexible curve, and then by Proposition 3.10, it can be degenerated to a sum of rational curves, so C' is algebraically equivalent to an integral sum of rational curves in Y , the rest is simply pushing forward back to X . □

3.15. Remark. As suggested by Professor János Kollár, one can prove the main theorem directly on X , by smoothing $C \cup C_1 \cup C_2 \dots \cup C_m$ to a curve C' with $H^1(C', T_X|_{C'}) = 0$ and then use the same argument above for the natural forgetful map

$$\bar{M}_{g,0}(X, \beta) \rightarrow \bar{M}_{g,0}$$

which is again proper and surjective—this will be discussed in [8].

3.16. Remark. As suggested by Professor Claire Voisin, based upon our result about algebraic equivalence, one can actually prove that all curves on X are rationally equivalent to a \mathbb{Z} -linear combination of rational curves, by using a construction of Professor János Kollár, see [8] for detail.

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